Potential Flow Formulation of Parker's Unsteady Solar Wind Model and Nonlinear Stability Aspects Near the Parker Sonic Critical Point

Bhimsen K. Shivamoggi*

California Institute of Technology
Pasadena, CA 91125

Abstract

The purpose of this paper is to present the first ever systematic theoretical formulation to address the long-standing issue of regularization of the singularity associated with the *Parker sonic critical point* in the linear perturbation problem for Parker's unsteady solar wind model. This is predicated on the necessity to go outside the framework of the linear perturbation problem and incorporate the dominant nonlinearities in this dynamical system. For this purpose, a whole new theoretical formulation of Parker's unsteady solar wind model based on the *potential flow theory* in ideal gas dynamics is given, which provides an appropriate optimal theoretical framework to accomplish this task. The stability of Parker's steady solar wind solution is shown to extend also to the neighborhood of the Parker sonic critical point by going to the concomitant nonlinear problem.

*Permanent Address: University of Central Florida, Orlando, FL, 32816

1 Introduction

The solar wind is a hot tenuous magnetized plasma outflowing continually from the sun which carries off a huge amount of angular momentum from the sun while causing only a negligible mass loss. The bulk of the solar wind is known to emerge from coronal holes (Sakao et al. [1]) and to fill the heliosphere (Dialynas et al. [2]). Coronal heating along with high thermal conduction is believed to be the cause of weak to moderate-speed solar wind. But some additional acceleration mechanism operating beyond the coronal base seems to be needed for high-speed solar wind (Parker [3] [4]). Parker [3] gave an ingenious model to accomplish this by continually converting the thermal energy into kinetic energy of the wind and accelerating the latter from subsonic to supersonic speeds. The various physical properties in the solar wind have been confirmed by *in situ* observations (Meyer-Vernet [5]). The Parker Solar Probe (Shivamoggi [6]) has been collecting a lot of significant information on the conditions in the solar corona (Fisk and Casper [7], Bowen et al. [8], and others) some of which were at variance with previous belief (like the coupling of the solar wind with solar rotation (Kasper et al. [9]), which was shown (Shivamoggi [10]) to cause enhanced angular momentum loss from the sun).

Parker's steady-solar wind solution is peculiar,

- in being the one solution that describes a smooth acceleration of the solar wind through the sonic conditions at the *Parker sonic critical point*, given by $r = r_* = GM_S/2a^2$, G being the gravitational constant, M_S is the mass of the sun, and a is the speed of sound;
- in corresponding to a special boundary condition prescribing the pressure to decrease away from the sun to zero at infinity in the interstellar space.

On the other hand, solar wind observations (Schrijver [11]) indicated that the large-scale behavior of the solar wind, on the average, its local noisiness (Feldman et al. [12]) notwithstanding, is apparently close to Parker's solar wind solution. This indicates that Parker's solar wind solution exhibits a certain robustness and an ability to sustain itself against any small perturbations acting on this system. Parker [13] therefore proposed that his solution possesses an intrinsic stability like a "*stable attractor*" of this dynamical system (Cranmer and Winebarger [14]). So, any deviations in flow variables from Parker's solar wind solution, Parker [13] argued, would be convected out by the wind flow and damped out.

This poses the stability of Parker's solar wind solution as an important issue, though still not completely resolved. This issue was investigated by Parker [15] via formal considerations of the dynamical equations governing the solar wind flow. Parker [15] advocated that the stability of the flow in the *subcritical* region inside the Parker sonic critical point may be considered by approximating the solar corona in this region by a static atmosphere on the grounds that no intrinsic shear-flow instabilities may be generated in the corona during its expansion in this region[1]. Shivamoggi [17] followed up on Parker's proposition for the subcritical region, and gave a systematic analytical development of this issue, by posing a Sturm-Liouville problem for the linearized perturbations about Parker's solar wind solution, to demonstrate its intrinsic stability.

On the other hand, Parker [15], Carovillano and King [18], and Jockers [19] initiated the investigation of stability of Parker's solar wind solution with respect to linearized perturbations by

[1] This is compatible with the absence of coronal-flow shear in the spherically symmetric flow situation posited in Parker's solar wind model [3], which would otherwise become a free-energy source of these shear-flow instabilities (Shivamoggi [16]).

including solar wind flow in the basic state and found that the singularity at the Parker sonic critical point makes this linear perturbation problem ill-posed. This precludes well-behaved solutions of the linear perturbation problem in the *transonic* flow region (where the wind flow-speed is near the speed of sound in the gas) near the Parker sonic critical point.

We wish to point out that a regularization of this singularity necessitates going outside of the framework of the linear perturbation problem and incorporating the dominant nonlinearities in this dynamical system (akin to the situation in *transonic aerodynamics* (Shivamoggi [20])). The straightforward unsteady version of Parker's solar wind model used so far for stability considerations lends a rather cumbersome mathematical approach toward this objective. The purpose of this paper is to present a whole new theoretical formulation of Parker's unsteady solar wind model based on the *potential flow theory* in ideal gas dynamics, which provides an optimal theoretical framework to analyze various aspects of Parker's unsteady solar wind model in general, and regularization of the singularity at the Parker sonic critical point by going to the concomitant nonlinear problem.

2 Potential-Flow Formulation of Parker's Unsteady Solar Wind Model

Consider an ideal gas flow in the presence of a central gravitating point mass representing the sun. The solar wind is represented by a spherically symmetric flow so the flow variables depend only on the distance r from the sun and time t, and the flow velocity is taken to be only in the radial direction.

The equations expressing the conservation of mass and momentum balance for the ideal gas flow constituting the solar wind are (in usual notations),

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 v\right) = 0 \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) = -\frac{\partial p}{\partial r} - \frac{dU}{dr} \tag{2}$$

where U is the gravitational potential associated with the sun (of mass M_S),

$$U = -\frac{GM_S}{r} \tag{3}$$

We assume the ideal gas flow under consideration to be modeled by a *potential flow*, so we have

$$v = \frac{\partial \Phi}{\partial r} \tag{4}$$

Furthermore, we assume for analytical simplicity that the gas flow occurs under *isothermal* conditions, so

$$p = a^2 \rho \tag{5}$$

where a is the constant speed of sound in the gas[2]. In the same vein, we assume that the flow variables as well as their derivatives vary continuously so there are no shocks occurring anywhere in the region under consideration.

[2]SOHO observations (Cho et al. [21]) indicated that the solar wind expands isothermally to considerable distances.

Using (4) and (5), equations (1) and (2) become

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r}\right) + \frac{1}{\rho}\frac{\partial \Phi}{\partial r}\frac{\partial \rho}{\partial r} = 0 \tag{6}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial r}\right) + \frac{\partial \Phi}{\partial r}\frac{\partial^2 \Phi}{\partial r^2} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial r} - \frac{dU}{dr}. \tag{7}$$

Equation (7) may be rewritten as

$$-\frac{1}{a^2}\frac{\partial \Phi}{\partial r}\left(\frac{\partial^2 \Phi}{\partial t \partial r} + \frac{\partial \Phi}{\partial r}\frac{\partial^2 \Phi}{\partial r^2}\right) - \frac{1}{a^2}\frac{\partial \Phi}{\partial r}\frac{\partial U}{\partial r} = \frac{1}{\rho}\frac{\partial \Phi}{\partial r}\frac{\partial \rho}{\partial r}. \tag{8}$$

On the other hand, the Bernoulli integral of equation (7),

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial r}\right)^2 + \int \frac{dp}{\rho} + U = const \tag{9}$$

gives,

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{1}{a^2}\left(\frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial \Phi}{\partial r}\frac{\partial^2 \Phi}{\partial t \partial r}\right) \tag{10}$$

Using equations (8) and (10), equation (6) leads to the equation governing the potential flows of an ideal gas constituting the solar wind,

$$\left[a^2 - \left(\frac{\partial \Phi}{\partial r}\right)^2\right]\frac{\partial^2 \Phi}{\partial r^2} + \frac{2a^2}{r}\frac{\partial \Phi}{\partial r} = \frac{\partial^2 \Phi}{\partial t^2} + 2\frac{\partial \Phi}{\partial r}\frac{\partial^2 \Phi}{\partial t \partial r} + \frac{\partial \Phi}{\partial r}\frac{dU}{dr}. \tag{11}$$

Equation (11) provides an optimal theoretical framework to extrapolate the Parker solar wind model to unsteady situations and investigate the long-standing issue of stability of the Parker steady solar wind solution.

3 Parker Steady Solar Wind Model

For a steady wind flow, equation (11) describes Parker's solar wind model [3],

$$\left[a^2 - \left(\frac{d\Phi}{dr}\right)^2\right]\frac{d^2 \Phi}{dr^2} + \frac{2a^2}{r^2}(r - r_*)\frac{d\Phi}{dr} = 0 \tag{12}$$

where $r = r_* \equiv GM_S/2a^2$ locates the *Parker sonic critical point.*

Equation (12) gives a physically acceptable smooth solution (Parker [3]),

$$\left[\frac{d\Phi/dr}{a}\right]^2 - \log\left[\frac{d\Phi/dr}{a}\right]^2 = 4\log\left(\frac{r}{r_*}\right) + 4\left(\frac{r}{r_*}\right) - 3 \tag{13}$$

which complies with the smoothness condition at the Parker sonic critical point,

$$r = r_* : \quad v = a. \tag{14}$$

4 Linear Perturbation Problem for Parker's Solar Wind Model

We assume solutions of time-dependent perturbations (denoted by subscript 0) to be of the form,

$$\Phi(r,t) = \phi_0(r) + \epsilon \phi_1(r,t), \quad \epsilon \ll 1 \tag{15}$$

and assume the perturbations characterized by the small parameter ϵ to be small. Equation (11) then yields for the basic state,

$$\left[a^2 - \left(\frac{d\phi_0}{dr}\right)^2\right] \frac{d^2\phi_0}{dr^2} + \frac{2a^2}{r^2}(r - r_*)\frac{d\phi_0}{dr} = 0 \tag{16}$$

which represents Parker's steady solar wind model given by equation (12), and for the linearized perturbations,

$$\left[a^2 - \left(\frac{d\phi_0}{dr}\right)^2\right] \frac{\partial^2\phi_1}{\partial r^2} + \left[-2\frac{d\phi_0}{dr}\frac{d^2\phi_0}{dr^2} + 2a^2(r - r_*)\right]\frac{\partial\phi_1}{\partial r} = 2\frac{d\phi_0}{dr}\frac{\partial^2\phi_1}{\partial t\partial r} + \frac{\partial^2\phi_1}{\partial t^2}. \tag{17}$$

We consider the subcritical region, where

$$\left[a^2 - \left(\frac{d\phi_0}{dr}\right)^2\right] > 0, \quad r < r_* \tag{18}$$

and assume normal-mode solutions of the form,

$$\phi_1(r,t) = \hat{\phi}_1(r)e^{-i\omega t}. \tag{19}$$

Equation (17) then gives

$$\frac{d^2\hat{\phi}_1}{dr^2} + \left[-\frac{d^2\phi_0/dr^2}{d\phi_0/dr} - 2\frac{(d\phi_0/dr)(d^2\phi_0/dr^2)}{a^2 - (d\phi_0/dr)^2} + 2i\omega\frac{d\phi_0/dr}{a^2 - (d\phi_0/dr)^2}\right]\frac{d\hat{\phi}_1}{dr} + \omega^2\left[\frac{1}{a^2 - (d\phi_0/dr)^2}\right]\phi_1 = 0. \tag{20}$$

Equation (20) may be written as the *Sturm-Liouville* equation,

$$\frac{d}{dr}\left[f(r)\frac{d\hat{\phi}_1}{dr}\right] + \omega^2 g(r)\hat{\phi}_1 = 0, \quad r_S < r < r_* \tag{21}$$

where

$$f(r) \equiv \left[\frac{a^2 - (d\phi_0/dr)^2}{d\phi_0/dr}\right] e^{2i\omega\int_{r_S}^{r} \frac{d\phi_0/dr}{a^2 - (d\phi_0/dr)^2} dr}$$

$$g(r) \equiv \frac{1}{d\phi_0/dr} e^{2i\omega\int_{r_S}^{r} \frac{d\phi_0/dr}{a^2 - (d\phi_0/dr)^2} dr}$$

r_S being sun's radius.

Taking the complex conjugate of equation (21) we have

$$\frac{d}{dr}\left[\bar{f}(r)\frac{d\bar{\hat{\phi}}_1}{dr}\right]+\omega^2\hat{(g)}(r)\bar{\hat{\phi}}_1=0, r_S<r<r_* \tag{22}$$

If ω is pure imaginary, $\omega = i\Omega$, we obtain from equations (21) and (22),

$$-\int_{r_S}^{r} f(r)\left|\frac{d\hat{\phi}_1}{dr}\right|^2 dr-\Omega^2\int_{r_S}^{r} g(r)\left|\hat{\phi}_1\right|^2 dr=0, \quad r_S<r<r_* \tag{23}$$

where we have taken the perturbations or their gradients to vanish at the coronal base $r = r_S$, and $f(r)$ and $g(r)$ now become

$$f(r)=\left[\frac{a^2-(d\phi_0/dr)^2}{d\phi_0/dr}\right]e^{-2\Omega\int_{r_S}^{r}\frac{d\phi_0/dr}{a^2-(d\phi_0/dr)^2}dr}>0$$

$$g(r)=\frac{1}{d\phi_0/dr}e^{-2\Omega\int_{r_S}^{r}\frac{d\phi_0/dr}{a^2-(d\phi_0/dr)^2}dr}>0.$$

Equation (23) is impossible to satisfy, so ω is real[3], and the Parker solar wind solution is linearly stable in the subcritical region.

It is to be noted, as previously mentioned by Parker [15], Carovillano and King [18] and Jockers [19], that the linearized perturbation problem, described by equation (20), exhibits a singularity at the Parker sonic critical point given by (14)[4]. Consequently, the above linearized development, which is valid in the subcritical region, becomes ill-posed and breaks down near the Parker sonic critical point. This drawback may be remedied via a proper treatment of the transonic flow region around the Parker sonic critical point. This necessitates going outside the linearized framework and adopting the nonlinear formulation (akin to the situation in *transonic aerodynamics* (Shivamoggi [20])). This task can be accomplished in an expeditious way by using the potential-flow formulation, namely equation (11), given in this paper.

5 Nonlinear Perturbation Problem for the Parker Solar Wind Model

Equation (11) governing the potential flows of an ideal gas constituting the solar wind may be rewritten as,

$$\left[a^2-\left(\frac{\partial\Phi}{\partial r}\right)^2\right]\frac{\partial^2\Phi}{\partial r^2}+\frac{2a^2}{r^2}(r-r_*)\frac{\partial\Phi}{\partial r}=\frac{\partial^2\Phi}{\partial t^2}+2\frac{\partial\Phi}{\partial r}\frac{\partial^2\Phi}{\partial t\partial r} \tag{24}$$

In order to treat the region near the Parker sonic critical point, described by $\partial\Phi/\partial r \approx a$, we follow the treatment of *thin airfoil in transonic flows* (Cole and Messiter [24]), and put, following *method of multiple scales* (Shivamoggi [25]),

[3]In non-dissipative systems (like the one under consideration) the transition from stability to instability may be expected to occur via a marginal state exhibiting oscillatory motions (Eddington [22], see also Chandrasekhar [23]).

[4]It may be mentioned, as Parker [15] pointed out, that this coincidence will not hold for more general non-isothermal cases.

$$
\frac{\partial \Phi}{\partial r} = a\left(1 + \epsilon \frac{\partial \phi_1}{\partial r}\right),
$$
$$
r = r_*(1+\epsilon x), \quad \tilde{t} = \epsilon t, \quad \epsilon \ll 1 \tag{25}
$$

where ϵ is a small parameter characterizing the deviation of the flow speed from the speed of sound in the gas. The slow (or shrunken) time scale $\tilde{t}$ characterizes the slowly varying dynamics under the influence of gravitational choking operational near the Parker sonic critical point. Equation (24)then yields

$$
\frac{1}{r_*}\frac{\partial \phi_1}{\partial x}\frac{\partial^2 \phi_1}{\partial x^2} - x\frac{\partial \phi_1}{\partial x} = -\frac{r_*}{a}\frac{\partial^2 \phi_1}{\partial x \partial \tilde{t}}. \tag{26}
$$

Putting further,

$$
u_1 \equiv \frac{\partial \phi_1}{\partial x}, \quad \tau \equiv \frac{a}{r_*^2}\tilde{t} \tag{27}
$$

equation (26) becomes

$$
\frac{\partial u_1}{\partial \tau} + u_1\frac{\partial u_1}{\partial x} = r_* x u_1. \tag{28}
$$

In order to determine a solution of this nonlinear hyperbolic equation, note first that the characteristics of equation (28) are given by

$$
C: \quad \frac{d\tau}{d\xi} = 1, \quad \frac{dx}{d\xi} = u_1 \tag{29}
$$

Equation (28) then reduces to the following ordinary differential equation,

$$
\frac{du_1}{d\xi} = r_* x(\xi) u_1(\xi), \quad \text{along C.} \tag{30}
$$

Equation (29) yields the solution,

$$
\tau = \xi, \quad x(x_0, \tau) = f(x_0, \tau) \tag{31}
$$

where,

$$
x_0 \equiv x(x_0, 0).
$$

Using (31), equation (30) yields the solution,

$$
u_1(x, \tau) = u_{1_0} e^{r_* \int_0^\tau f(x_0, S) dS}. \tag{32}
$$

where,

$$
u_{1_0} \equiv u_1(x, 0).
$$

Introduce,

$$
\psi(\tau) \equiv r_* \int_0^\tau f(x_0, S) dS \tag{33}
$$

which yields, on using (31),

$$
\frac{d\psi}{d\tau} = r_* f(x_0, \tau) = r_* x. \tag{34}
$$

(33) and (34) imply the initial conditions,

$$\tau = 0 : \quad \psi = 0, \quad \frac{d\psi}{d\tau} = r_* x_0. \tag{35}$$

Furthermore, on using equations (29), (32), and (34), we have

$$\frac{d^2\psi}{d\tau^2} = r_* \frac{dx}{d\tau} = r_* u_1 = r_* u_1 e^{\psi} \tag{36}$$

from which, we obtain

$$\frac{d\psi}{d\tau} = \sqrt{2 r_* u_{1_0}} e^{\psi/2}. \tag{37}$$

Equation (37) yields the solution,

$$e^{-\psi/2} = 1 - \frac{1}{2} r_* x_0 \tau. \tag{38}$$

Using (38), (32) and (33) give

$$u_1(x,\tau) = \frac{u_{1_0}}{(1 - r_* x_0 \tau/2)^2}. \tag{39}$$

Furthermore, (35) and (37) yield

$$u_{1_0} = \frac{r_* x_0^2}{2} \tag{40}$$

and (39) becomes

$$u_1(x,\tau) = \frac{r_* x_0^2/2}{(1 - r_* x_0 \tau/2)^2}. \tag{41}$$

Using (41), equation (29) yields,

$$x(x_0,\tau) = \frac{x_0}{1 - r_* x_0 \tau/2} \tag{42}$$

from which, we obtain

$$x_0 = \frac{x}{1 + r_* x \tau/2}. \tag{43}$$

Using (43), (41) becomes

$$u_1(x,\tau) = \frac{1}{2} r_* x^2. \tag{44}$$

(44) implies that the dynamics in the nonlinear perturbation problem near the Parker sonic critical point is essentially frozen in time. Physically this seems to be traceable to the gravitational choking (described by the term on the right in equation (28)) operational in the nonlinear hyperbolic dynamics near the Parker sonic critical point. Indeed, in the time-independent limit, equation (28) becomes

$$u_1(u_{1x} - r_* x) = 0 \tag{45}$$

from which, (ruling out the trivial solution $u \equiv 0$),

$$u_1(x,\tau) = \frac{1}{2} r_* x^2 \tag{46}$$

in agreement with (44).

6 Discussion

Contrary to the assumptions made in the theoretical models, the solar wind is, in reality, far from being steady and structureless, as revealed by spatial and temporal variabilities apparent in *in situ* observations of the solar wind. Nonetheless, Parker's solar wind solution has been found to provide an excellent first-order approximation to the large-scale behavior, on the average, of the solar wind. This indicates it has a certain robustness and an ability to sustain itself against any small perturbations acting on this system. This poses stability of Parker's solar wind solution as an important issue, though still not completely resolved. Previous investigations ([15], [18], [19]) of stability of Parker's solar wind solution with respect to linearized perturbations were plagued by the singularity at the Parker sonic critical point, where the wind flow equals the speed of sound in the gas. This paper seeks to regularize this singularity by going outside the framework of the linear perturbation problem, and incorporating the dominant nonlinearities in this dynamical system. This is implemented by introducing a whole new theoretical formulation of Parker's solar wind model based on the *potential flow theory* in ideal gas dynamics, which provides an appropriate optimal theoretical framework for this purpose. The stability of Parker's solar wind solution is shown to extend also to the neighborhood of the Parker sonic critical point by going to the concomitant nonlinear problem.

7 Acknowledgments

This work was carried out during my sabbatical leave at California Institute of Technology. My thanks are due to Professor Shrinivas Kulkarni for his enormous hospitality and valuable discussions. I am thankful to late Professor Eugene Parker for his helpful advice and suggestions on the solar wind problem.

References

[1] T. Sakao et al.: *Science* **318**, 1585, (2007).

[2] K. Dialynas et al.: *Nature Astronomy* **1**, 0115, (2017).

[3] E. N. Parker: *Astrophys. J.* **128**, 664, (1958).

[4] E. N. Parker: *Space Sci. Rev.* **4**, 666, (1965).

[5] N. Meyer-Vernet: *Basics of the Solar Wind*, Cambridge Univ. Press, (2007).

[6] B. K. Shivamoggi: *Amer. Scientist*, **112**, 108, (2024).

[7] L. A. Fisk and J. C. Kasper: *Astrophys. J. Lett.* **894**, L4, (2020).

[8] T. A. Bowen et al.: *Phys. Rev. Lett.* **129**, 165101, (2022).

[9] J. C. Kasper et al.: *Nature* **576**, 228, (2019).

[10] B. K. Shivamoggi: *Phys. Plasmas* **27**, 012902 , (2020).

[11] C. J. Schrijver: *Solar and Stellar Magnetic Activity*, Cambridge Univ. Press, (2000).

[12] W. C. Feldman et al.: in *The Solar Output and its Variations* (Ed. O. R. White), Colorado Associated University Press, (1977).

[13] E. N. Parker: *Astrophys. J.* **132**, 821, (1960).

[14] S. R. Cranmer and A. R. Winebarger: *Ann. Rev. Astron. Astrophys.* **57**, 157, (2019).

[15] E. N. Parker: *Astrophys. J.* **143**, 32, (1966).

[16] B. K. Shivamoggi: *Stability of Parallel Gas Flows*, Ellis Horwood, (1986).

[17] B. K. Shivamoggi: *Astrophys. J.* **944**, 96, (2023).

[18] R. L. Carovillano and J. H. King: *Astrophys. J.* **145**, 426, (1966).

[19] K. Jockers: *Solar Phys.* **3**, 603, (1968).

[20] B. K. Shivamoggi: *Introduction to Theoretical and Mathematical Fluid Dynamics*, Wiley, (2023).

[21] B. H. Cho et al.: *Phys. Rev. Lett.* **121**, 075101, (2018).

[22] A. S. Eddington: *The Internal Constitution of stars*, Cambridge Univ. Press (1926).

[23] S. Chandrasakhar: *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, (1961).

[24] J. D. Cole and A. F. Messiter: *Z. Angew. Math. Phys.* **8**, 1, (1957).

[25] B. K. Shivamoggi: *Perturbation Methods for Differential Equations*, Birkhauser, (2003).